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STATE OF STRESS AND STRAIN OF A SMALL NEIGHBOURHOOD OF THE APEX OF A WEDGE FOR A PHYSICAL NON-LINEARITY AND DIFFERENT BOUNDARY CONDITIONS*

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Three plane strain problems of a small neighbourhood of the apex of a physically non-linear incompressible wedge are investigated by the Cherepanov-Rice-Hutchinson method using a non-linear differential equation for the Airy stress function: Problem 1 - one face is free, and a smooth contact condition is given on the other; Problem 2 - one face if free, and a flexible inextensible cover plate is glued to the other; Problem 3 - one fact is free and the condition of adhesion to a stiff flat stamp is given on the other. Numerical results are presented that illustrate the influence of the degree of non-linearity of the governing relationships and the wedge aperture angle on the solution. The method is also applied to the stream function which enables us to formulate an analogy between different plane problems and affords the possibility of extending it to the axisymmetric case. In many problems of the mechanics of a deformable solid, the investigation of the asymptotic form of the solution near an angular point of a domain occupied by a body plays a fundamental role. In the elastic case this question has been studied quite broadly and an extensive literature exists. The situation is more complicated if the governing relationships are non-linear. The majority of papers deal only with the case of a crack. This paper attempts to fill this gap somewhat.

1. We consider the problem of the equilibrium of a wedge with apperture angle α from a material subjected to the law

$$\varepsilon_u = A\sigma_u{}^m, \quad \varepsilon_{kk} = 0, \quad S_{ij} = \sigma_u \varepsilon_u^{-1} \varepsilon_{ij}, \quad A, m = \text{const}, \quad m \geqslant 1 \tag{1.1}$$

$$\sigma_u = 6^{-1/2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6\sigma_{12}{}^2 + 6\sigma_{23}{}^2 + 6\sigma_{31}{}^2]^{1/2}$$

$$\varepsilon_u = 6^{-1/2} [(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 + 6\varepsilon_{12}{}^2 + 6\varepsilon_{23}{}^2 + 6\varepsilon_{31}{}^2]^{1/2}$$

Here S_{ij} are the components of the stress deviator in a certain orthonormalized basis, σ_u is the stress intensity, ε_{ij} are the components of the strain or strain rate tensor depending on the specific model: if the problem of non-linear steady creep is considered, then ε_{ij} is the rate, if an elastic-plastic tension-compression diagram is described by (1.1), generally speaking, then ε_{ij} are tensor components of small strain. There is no need to make the physical meaning of ε_{ij} specific; by virtue of a well-known analogy the fundamental equations are written identically, and consequently, we will henceforth call ε_{ij} the strain for brevity, and ε_u the strain intensity.

We assume the strain to be planar. In polar coordinates with centre of the wedge apex we have

$$S_r = -S_\vartheta = \frac{\sigma_u}{\varepsilon_u} \varepsilon_r, \quad S_{r\vartheta} = \frac{\sigma_u}{\varepsilon_u} \varepsilon_{r\vartheta}, \quad \sigma = \sigma_z = \frac{\sigma_r + \sigma_\vartheta}{2} \tag{1.2}$$

$$\sigma_u = [{}^1/_4 (\sigma_r - \sigma_\vartheta)^2 + \sigma_{r\vartheta}^2]^{1/2}, \quad \varepsilon_u = [\varepsilon_r{}^2 + \varepsilon_{r\vartheta}^2]^{1/2}$$

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The equilibrium equations and Cauchy relationships

$$\sigma_{r,r} + \frac{1}{r}(\sigma_r - \sigma_\vartheta) + \frac{1}{r}\sigma_{r\vartheta,\vartheta} = 0 \tag{1.3}$$

$$\sigma_{r\vartheta,r} + \frac{1}{r}\sigma_{\vartheta,\vartheta} + \frac{2}{r}\sigma_{r\vartheta} = 0$$

$$\varepsilon_r = u_{r,r}, \quad \varepsilon_\vartheta = \frac{1}{r}u_{\vartheta,\vartheta} + \frac{1}{r}u_r \tag{1.4}$$

$$\varepsilon_{r\vartheta} = \frac{1}{2}\left(u_{\vartheta,r} - \frac{1}{r}u_\vartheta + \frac{1}{r}u_{r,\vartheta}\right)$$

should be satisfied everywhere in the domain (the subscript without a comma corresponds to a vector or tensor component, while the subscript after a comma dentoes the operation of differentiation with respect to the corresponding coordinate).

We introduce the Airy stress function Φ by means of the usual formulas

$$\sigma_r = \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\vartheta\vartheta}, \quad \sigma_\vartheta = \Phi_{,rr}, \quad \sigma_{r\vartheta} = \frac{1}{r^2}\Phi_{,\vartheta} - \frac{1}{r}\Phi_{,r\vartheta} \tag{1.5}$$

Expressing the strain components from (1.1), (1.2), (1.5) in terms of Φ and substituting these expressions into the single compatibility condition for the plane problem

$$\frac{1}{r}(r\varepsilon_\vartheta)_{,rr} + \frac{1}{r^2}\varepsilon_{r,\vartheta\vartheta} - \frac{1}{r}\varepsilon_{r,r} - \frac{2}{r^2}(r\varepsilon_{r\vartheta,\vartheta})_{,r} = 0 \tag{1.6}$$

we obtain a certain partial differential equation in the Airy function whose solution under given boundary conditions determines the stress and strain fields completely.

Since only the behaviour of the solution in a small neighbourhood of the apex is of interest, we will seek the Airy function in the form

$$\Phi(r, \vartheta) = Qr^s\varphi(\vartheta) \tag{1.7}$$

where Q and s are unknown real constants, and φ is an unknown function of the variable ϑ. This latter representation is used extensively in investigating the solution of problems near an angular point. It can be treated as the principal term of the asymptotic expansion in powers of the variable r. From the practical viewpoint, it simplifies the problem considerably by separating variables in the compatibility equation. The scheme taken with assumption (1.7) is called the Cherepanov-Rice-Hutchinson method from the names of the authors of /1-5/, who first used this method in problems of cracks in a medium being hardened according to a power law /6/.

Thus by assuming the validity of relationship (1.7), we have from (1.5), (1.1) and (1.2)

$$\sigma_{ij}(r, \vartheta) = Qr^{s-2}\bar\sigma_{ij}(\vartheta) \tag{1.8}$$

$$\bar\sigma_u = [{}^1/_4(\varphi'' + s(2-s)\varphi)^2 + (1-s)^2\varphi'^2]^{1/2}$$

$$\bar\sigma_r = s\varphi + \varphi'', \quad \bar\sigma_\vartheta = s(s-1)\varphi, \quad \bar\sigma_{r\vartheta} = (1-s)\varphi'$$

$$\varepsilon_{ij}(r, \vartheta) = AQ^m r^{m(s-2)}\bar\varepsilon_{ij}(\vartheta) \tag{1.9}$$

$$\bar\varepsilon_r = -\bar\varepsilon_\vartheta = \bar\sigma_u^{m-1}\cdot{}^1/_2(\varphi'' + s(2-s)\varphi), \quad \bar\varepsilon_{r\vartheta} = \bar\sigma_u^{m-1}(1-s)\varphi'$$

Evaluating the necessary derivatives of the strain tensor components and substituting them into the compatability condition (1.6), we arrive at the following fourth-order ordinary differential equations in the function $\varphi(\vartheta)$ defined in the segment $[0, \alpha]$ in which the constant s occurs as a parameter

$$\left\{\frac{d^2}{d\vartheta^2} - m(s-2)[m(s-2)+2]\right\}\{\bar\sigma_u^{m-1}\cdot{}^1/_2(\varphi'' + s(2-s)\varphi)\} + \tag{1.10}$$
$$2(s-1)[m(s-2)+1]\{\bar\sigma_u^{m-1}\varphi'\}' = 0$$

The natural requirement of finite strain energy in the neighbourhood of an apex of arbitrary radius ρ imposes the constraint

$$\int_0^\rho \varepsilon_{ij}\sigma_{ij}r\,dr < \infty; \quad s > \frac{2m}{m+1} \tag{1.11}$$

We now consider the boundary conditions. We will consider the wedge face $\vartheta = \alpha$ as load-free, which, taking account of (1.9), is written as

$$\varphi(\alpha) = \varphi'(\alpha) = 0 \tag{1.12}$$

We require satisfaction of one of the following three conditions on the other wedge face, and we define three problems in this connection.

Problem 1. A rigid stamp with a flat absolutely smooth base acts on the fact $\vartheta = 0$

$$u_\vartheta(r, 0) = \text{const}, \quad \sigma_{r\vartheta}(r, 0) = 0 \tag{1.13}$$

It is seen /7/ that (1.13) is equivalent to the condition

$$\varphi'(0) = \varphi'''(0) = 0 \tag{1.14}$$

Problem 2. A cover plate that does not resist bending but is rigid in tension is glued to the face $\vartheta = 0$

$$u_r(r, 0) = \text{const}, \quad \sigma_\vartheta(r, 0) = 0 \tag{1.15}$$

The equivalence of (1.15) to the following requirement results from (1.9):

$$\varphi(0) = \varphi''(0) = 0 \tag{1.16}$$

Problem 3. A flat rigid stamp adheres to the face $\vartheta = 0$

$$u_r(r, 0) = \text{const}, \quad u_\vartheta(r, 0) = C_1 r + C_2, \quad C_1, C_2 = \text{const} \tag{1.17}$$

The first equality in (1.17), the first relationship in (1.4) and (1.9) yield

$$\varphi''(0) + s(2 - s)\varphi(0) = 0 \tag{1.18}$$

Taking account of the second equality in (1.17), we have from the first and third Cauchy relationship (1.4) that $(2r\varepsilon_{r\vartheta})_{,r} - \varepsilon_{r,\vartheta} = 0$ for $\vartheta = 0$. Differentiating and substituting the necessary expressions (1.9), here, using (1.18), we obtain after some reduction

$$\varphi'''(0) - \varphi'(0)\,[4\,(m\,(s - 2) + 1)(1 - s) - s\,(2 - s)] = 0 \tag{1.19}$$

Thus, conditions (1.17) are equivalent to (1.18) and (1.19).

The left-hand sides of (1.10) and all the conditions (1.12), (1.14), (1.16), (1.18) and (1.19) are homogeneous functions in φ and the derivatives of φ with respect to ϑ; hence the problems allow non-trivial solutions only for particular values of the parameter s. Therefore, in addition to the function $\varphi(\vartheta)$ not identically equal to zero, the least real and simple eigennumber s satisfying the energy estimate (1.11) must be found.

The problems formulated can be solved by the method of adjustment as follows. By using the homogeneity, we obtain normalization by requiring $\varphi(0) = -1$ in problems 1 and 3 and $\varphi'(0) = -1$ in problem 2. We thus obtain the third condition at the left end of the segment of integration. Now giving some value to the parameter s and the fourth condition at zero, we can solve the Cauchy problem numerically by the Runge-Kutta method. At the point $\vartheta = \alpha$ we will here obtain certain numbers $a = \varphi(\alpha)$, $b = \varphi'(\alpha)$. Applying Newton's method to the function thus determined, it is easy to achieve $a = b = 0$ and thereby to satisfy condition (1.12). The inequality (1.11) and the requirement that the s found would actually be the least of the numbers satisfying it are confirmed directly.

2. On the basis of the above, a series of calculations was carried out. Problems about cracks from /4, 5/, obtained in the special case of $\alpha = \pi$ upon requirement of normalization were used as a test for the program. The results of the computations are represented partially in Figs.1-4.

Fig.1a shows graphs of the dependence of the singularity index s on the wedge α aperture angle for several m in problem 1. Since the conditions of a smooth stamp (1.14) are identical to the symmetry conditions, this can be utilized to check the calculations. Known particular results are isolated by points on the graphs. For $\alpha = \pi$ the formula /1-4/

$$s = (2m + 1)/(m + 1) \tag{2.1}$$

holds over the whole real range of variation of m.

The value $\alpha = \pi/2$ corresponds to the problem of the tension or compression of a half-plane by forces parallel to its boundary. Independently of m the stresses should obviously be bounded in this case; consequently, $s = 2$. The point extracted on the curve for $m = 3$ corresponds to the value $s = 1.775$ mentioned in /4/ as an illustration of the possiblity of solving the problem for $\alpha \neq \pi$ by the method utilized here. The curve for $m = 1$ can be obtained as a result of solving the algebraic equation. The range of variation of m is taken from tables /8/.

To answer the question of why s is the least of the numbers satisfying (1.11), numerous attempts to construct a solution with s less than in Fig.1a were made by using the same program, but for other starting parameters in addition to the reasoning of continuity during passages to the limit $\alpha \to \pi$, $\alpha \to \pi/2$, $m \to 1$. They were not successful: the Newton process either converged to the calculated values or diverged. Figs.1 b-d show graphs of the angular components of the local stress field. The numbers on the curves correspond to the components thus: $1 - \bar{\sigma}_u(\vartheta)$, $2 - \bar{\sigma}_{r\vartheta}(\vartheta)$, $3 - \bar{\sigma}_r(\vartheta)$, $4 - \bar{\sigma}_\vartheta(\vartheta)$. Case b corresponds to a homogeneous field for $\alpha = \pi/2$, $s = 2$, $\varphi = -\cos^2\vartheta$. There are graphs for $m = 3$, $\alpha = 3\pi/4$ (Fig.1c) with the normalization $\varphi(0) = 1$ in /4/. Fig.1d shows graphs for $m = 13$, $\alpha = 3\pi/4$. In this case $s = 1.933$, $\varphi''(0) = 1.083$.

Let us consider problem 2 (Figs.2 and 3). Fig.2 is analogous to Fig.1a. The isolated points for $\alpha = \pi$ correspond to values of s given by (2.1) /5/. This is the only case when the values of s in problems 1 and 2 agree. As α decreases, the eigennumber of problem 2 grows, and more rapidly than in problem 1. Finally, a certain critical value $\alpha = \alpha^*$ is reached when $s = 2$ and the stresses become bounded. This part of the figure is most interesting; it is reproduced on a larger scale.

Unlike problem 1 the magnitude of the critical angle α^* depends substantially on m. The greater the value of m the smaller α^* becomes. In the linear case the value of α^* is found easily from the equation $\tan 2\alpha^* = 2\alpha^*$. To determine α^* for $m \neq 1$ it is possible to set $s = 2$ at once in (1.10). Then the order of the equation is reduced to one, and by a chain of substitutions it is transformed into a Riccati equation and reduced to a Bessel equation for functions of the order of $\nu = (m - 1)/(2m)$. Execution of the reverse chain of substitutions results in a non-trivial relationship between φ', φ'' and three free constants containing modified Bessel functions of the first kind of orders ν and $\nu - 1$; consequently, the problem of finding α^* for which the boundary conditions can be satisfied is more complicated than the original one.

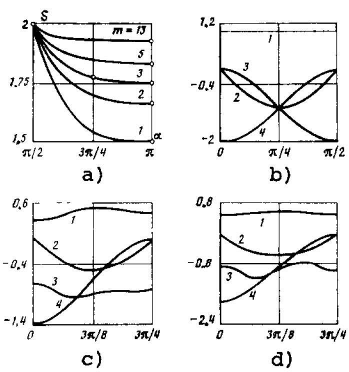

Fig.1

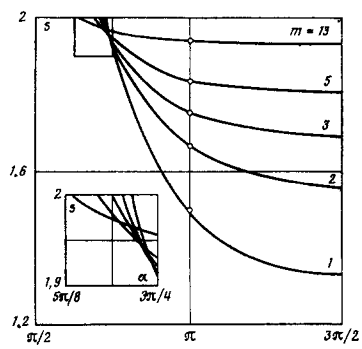

Fig.2

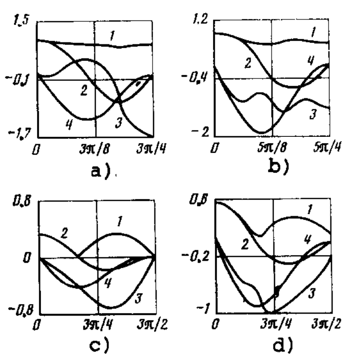

Fig.3

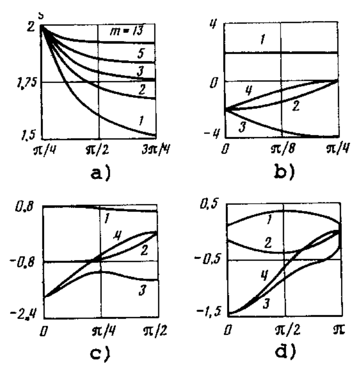

Fig.4

Fig.3 shows the angular local stress field components in the problem under consideration. The numbering of the curves is analogous to Fig.1. For $m = 1$ the solution is known. There are graphs for $\alpha = \pi$, $m = 3$ and $m = 13$ in /5/. The following cases are represented in Fig.3 a $- m = 13$, $s = 1.954$; b $- m = 7$, $s = 1.861$; c $- m = 1$, $s = 1.333$; d $- m = 2$, $s = 1.563$. The appearance of a break in the graph of $\bar{\sigma}_u(\vartheta)$ of part c forced us to confine ourselves to the range $\alpha \leqslant 3\pi/2$.

Unlike problems 1and 2, problem 3 is new in the non-linear case and consequently, more interesting. The incompressibility of the material and the non-linearity of the law of its behaviour (1.1) enabled us to hope for no stress oscillation near the angle and enabled us to investigate the asymptotic form by proceeding according to the preceding scheme. There was no heuristic reasoning relative to the quantity $s(\alpha)$ for $m > 1$. It turned out to be convenient to select $\alpha = 3\pi/4$ as the initial value.

The results of calculations are represented in Fig.4. Fig.4a is analogous to Figs.1a and 2a. It is seen that even for $\alpha = 3\pi/4$ the calculated values of s are close to the values given by (2.1) while as α tends to π they approach them still more. However, the case $\alpha = \pi$ is not achievable, the tangents to the graphs $\bar{\sigma}_r(\vartheta)$ and $\bar{\sigma}_u(\vartheta)$ become vertical at the point $\vartheta = \pi$ (Fig.4d, where $m = 4$, $s = 1.8$), it is required to seek the asymptotic form in a form different from (1.7). For $m = 1$, $\alpha = \pi$ conditions (1.12), (1.18) and (1.19) and normalization do not permit extraction of a unique solution.

For $\alpha = \pi/4$ all the curves of Fig.4a intersect at one point $s = 2$; therefore, as in problem 1, a single critical angle $\alpha^* = \pi/4$ exists for all m. The pattern of the stresses in this case is displayed in Fig.4b. The stress intensity (curve *1*) is constant and equal to two, Eq.(1.10) becomes linear, and the boundary value problem is solved exactly: $\varphi = \sin 2\vartheta - 1$. Such a function corresponds to a homogeneous stress field on areas perpendicular to the free face of the wedge and the deformation plane, the normal stresses are $\bar\sigma_n \equiv -4$, the tangential stresses are $\bar\sigma_\tau \equiv 0$ and on areas parallel to the free surface $\bar\sigma_n = \bar\sigma_\tau = 0$. Setting $m = 1, s = 2$ in the initial problem, it can be seen that the value of the critical angle found by the program is minimal. The case $m = 13, s = 1.933$ is shown in Fig.4c.

3. The asymptotic forms of the desired fields near an angular point can be constructed for an incompressible material by the following method that differs from the one examined in Sect.1. Substituting the first two Cauchy relations from (1.4) into the incompressibility condition, multiplying by r and grouping components we obtain a first-order partial differential equation

$$(ru_r)_{,r} + u_{\vartheta,\vartheta} = 0 \tag{3.1}$$

Without loss of generality, it can be satisfied by introducing the ordinary stream function $\Psi(r, \vartheta)$ such that

$$u_\vartheta = -\Psi_{,r}, \quad u_r = r^{-1}\Psi_{,\vartheta} \tag{3.2}$$

The strain components are expressed in terms of Ψ thus

$$\begin{gathered}\varepsilon_r = -\varepsilon_\vartheta = -r^{-2}\Psi_{,\vartheta} + r^{-1}\Psi_{,r} \\ \varepsilon_{r\vartheta} = {}^1/_2(-\Psi_{,rr} + r^{-1}\Psi_{,r} + r^{-2}\Psi_{,\vartheta\vartheta}) \\ \varepsilon_u = [(r^{-1}\Psi_{,\vartheta r} - r^{-2}\Psi_{,\vartheta})^2 + {}^1/_4(-\Psi_{,rr} + r^{-1}\Psi_{,r} + r^{-2}\Psi_{,\vartheta\vartheta})^2]^{1/2}\end{gathered} \tag{3.3}$$

We rewrite the equilibrium Eq.(1.3) by extracting from the operators of the left-hand side the deviator and global components

$$\begin{gathered}S_{,rr} + \frac{1}{r}S_{r\vartheta,\vartheta} + \frac{1}{r}(S_r - S_\vartheta) + \sigma_{,r} \equiv B_1[S_{ij}] + \sigma_{,r} = 0 \\ S_{r\vartheta,r} + \frac{1}{r}S_{\vartheta,\vartheta} + \frac{2}{r}S_{r\vartheta} + \frac{1}{r}\sigma_{,\vartheta} \equiv B_2[S_{ij}] + \frac{1}{r}\sigma_{,\vartheta} = 0\end{gathered} \tag{3.4}$$

Differentiating the first equation in (3.4) with respect to ϑ, we obtain

$$\sigma_{,r\vartheta} = -B_{1,\vartheta} = -S_{r,r\vartheta} - \frac{1}{r}S_{r\vartheta,\vartheta\vartheta} - \frac{1}{r}(S_{r,\vartheta} - S_{\vartheta,\vartheta}) \tag{3.5}$$

Multiplying the second equation in (3.4) by r, differentiating it with respect to r and using (3.5), we will have

$$\begin{gathered}0 = B_2 + rB_{2,r} - B_{1,\vartheta} = 3S_{r\vartheta,r} + \\ \frac{1}{r}S_{\vartheta,\vartheta} + rS_{r\vartheta,rr} + S_{\vartheta,\vartheta r} - S_{r,r\vartheta} - \frac{1}{r}S_{r\vartheta,\vartheta\vartheta} - \frac{1}{r}S_{r,\vartheta}\end{gathered} \tag{3.6}$$

The right-hand side of this last equation contains derivatives only of the stress deviator components which are expressed from the governing relationships (1.1) and (1.2) in terms of the strain, and this in turn in terms of Ψ from (3.3). Therefore, to find all the unknown quantities, it is again required to seek just one function from a fourth-order equation under certain boundary conditions.

By analogy with (1.7), we seek the stream function Ψ in the form

$$\Psi(r, \vartheta) = Pr^t\psi(\vartheta), \quad P, t = \text{const} \tag{3.7}$$

We then obtain from (3.7), (3.2) and (3.3)

$$\begin{gathered}u_i = Pr^{t-1}\bar u_i(\vartheta), \quad \bar u_r = \psi', \ \bar u_\vartheta = -t\psi \\ \varepsilon_{ij} = Pr^{t-2}\bar\varepsilon_{ij}(\vartheta), \quad \bar\varepsilon_r = -\bar\varepsilon_\vartheta = (t-1)\psi' \\ \bar\varepsilon_{r\vartheta} = {}^1/_2(\psi'' + t(2-t)\psi), \quad \bar\varepsilon_u = [(t-1)^2\psi'^2 + \\ {}^1/_4(\psi'' + t(2-t)\psi)^2]^{1/2}\end{gathered} \tag{3.8}$$

Solving (1.1) for the stress and using (3.8), we will have

$$S_{ij} = K\varepsilon_u^{\mu-1}\varepsilon_{ij}, \quad \mu = m^{-1}, \quad K = A^{-\mu} \tag{3.9}$$

$$\begin{gathered}S_r = -S_\vartheta = W(r, \vartheta)\bar\varepsilon_r, \quad S_{r\vartheta} = W\bar\varepsilon_{r\vartheta} \\ W(r, \vartheta) \equiv KP^\mu r^{\mu(t-2)}\bar\varepsilon_u^{\mu-1}\end{gathered} \tag{3.10}$$

Noting that

$$W_{,r} = r^{-1}\mu(t-2)W, \quad W_{,rr} = r^{-2}\mu(t-2)[\mu(t-2)-1]W \tag{3.11}$$
$$W_{,\vartheta} = {}^1/_2(\mu-1)E'E^{-1}W, \quad W_{,\vartheta\vartheta} = {}^1/_2(\mu-1)[E''E^{-1} + {}^1/_2(\mu-3)E'^2E^{-2}]W$$
$$W_{,r\vartheta} = {}^1/_2 r^{-1}\mu(\mu-1)(t-2)E'E^{-1}W, \quad E \equiv \bar{\varepsilon}_u^2 = (t-1)^2\psi'^2 - \chi^2, \quad \chi \equiv \bar{\varepsilon}_{r\vartheta}$$

and evaluating the necessary derivatives of the stress deviator components and then substituting them into (3.6), we obtain a non-linear fourth-order ordinary differential equation in the function $\psi(\vartheta)$ containing t as a parameter

$$\mu(t-2)[\mu(t-2)+2]\chi - {}^1/_2(\mu-1)[E''E^{-1} + {}^1/_2(\mu-3)E'^2E^{-2}]\chi - \chi'(\mu-1)E'E^{-1} - \chi'' + (\mu-1)(1-t)E'E^{-1}\psi'(\mu(t-2)+1) + 2(1-t)(\mu(t-2)+1)\psi'' = 0 \tag{3.12}$$

Without expanding the notation we note that ψ'''' occurs in (3.12) in terms of E'' and χ''.

Let us obtain the boundary conditions. This is done most simply for the coupled stamp. In a coordinate system coupled rigidly to the stamp the conditions of total coupling along the faces $\vartheta = \alpha$ $u_r(r, \alpha) = 0$, $u_\vartheta(r, \alpha) = 0$ are equivalent by virtue of (3.8) to the requirement.

$$\psi(\alpha) = \psi'(\alpha) = 0 \tag{3.13}$$

If a stamp with a smooth flat base acts on the fact $\vartheta = 0$, the condition $u_\vartheta(r, 0) = 0$ is equivalent to $\psi(0) = 0$ and the requirement $\sigma_{r\vartheta}(r, 0) = 0$ means that

$$\psi''(0) + t(2-t)\psi(0) = 0 \tag{3.14}$$

Together this yields

$$\psi(0) = \psi''(0) = 0 \tag{3.15}$$

The derivation of the following condition turns out to be somewhat more complex. We assume that a flexible inextensible coverplate glued to the face $\vartheta = 0$ is loaded by a uniform pressure $p = \text{const}$. The equality $u_r(r, 0) = 0$ is equivalent to the following: $\psi'(0) = 0$. Differentiating the equality $\sigma_\vartheta(r, 0) = p$, we obtain that $\sigma_{\vartheta, r} \equiv S_{\vartheta, r} + \sigma_{,r} = 0$ for $\vartheta = 0$. Since it follows from (3.10) that $S_{\vartheta, r} = r^{-1}\mu(t-2)(1-t)\psi' W$, and $\sigma_{,r}$ is expressed in terms of the derivatives of the stress deviator components from the first equilibrium Eq. (3.4), then by using (3.10) and (3.11), we will have

$$\psi'''(0) - \psi'(0)[4(\mu(t-2)+1)(1-t) - t(2-t)] = 0 \tag{3.16}$$

Together with the first requirement this means that

$$\psi'''(0) = \psi'(0) = 0 \tag{3.17}$$

Finally, the last condition. The wedge face $\vartheta = 0$ is loaded by uniform pressure $\sigma_\vartheta(r, 0) = p$, $\sigma_{r\vartheta}(r, 0) = 0$. This is necessary and sufficient to satisfy (3.14) and (3.16).

The estimate of t from power considerations analogous to (1.11) results in the inequality

$$t > 2\mu/(\mu+1) \tag{3.18}$$

Apart from the renotation $\psi \mapsto \varphi$, $t \mapsto s$, $\mu \mapsto m$ the conditions (3.13), (3.15), (3.17), (3.14) and (3.16) agree with (1.12), (1.16), (1.14), (1.18) and (1.19), respectively. In exactly the same manner Eq. (3.12) can be reduced to the form (1.10) with the same renotation. Only the meaning of the quantities in all the formulas changes. This enables us to formulate the following analogy.

We assume that the boundary value problem (1.10) is solved with the conditions (1.18) and (1.19) at zero and (1.12) at $\vartheta = \alpha$, i.e., the eigenfunction $f(\vartheta)$ and eigennumber z satisfying (1.11) are found. Two mechanical problems are thereby actually solved. If f is treated as an angular component of the Airy function ($f \equiv \varphi$, $z \equiv s$), problem 3 of Sect. 1 is solved: formulas (1.8) and (1.9) completely define the state of stress and strain of the apex of a wedge from a material subject to the law $\varepsilon_u = A\sigma_u^m$ for the coupled face $\vartheta = 0$ and the free face $\vartheta = \alpha$. According to (1.9), (1.8) and (1.4) the desired fields behave thus:

$$u_i \sim r^{m(s-2)+1}, \quad \varepsilon_{ij} \sim r^{m(s-2)}, \quad \sigma_{ij} \sim r^{s-2}, \quad r \to 0 \tag{3.19}$$

If f is considered to be an angular component of the stream function ($f \equiv \psi$, $z \equiv t$) then by (3.8) and (3.10) the local solution of the problem of plane strain of the apex of a wedge from a material with a mirror-symmetric diagram is determined

$$\varepsilon_u = A\sigma_u^\mu, \quad \mu = m^{-1} \tag{3.20}$$

for the free face $\vartheta = 0$ and the coupled face $\vartheta = \alpha$. Here

$$u_i \sim r^{t-1}, \quad \varepsilon_{ij} \sim r^{t-2}, \quad \sigma_{ij} \sim r^{m(t-2)}, \quad r \to 0 \tag{3.21}$$

Apart from a change in the faces, the identical problem is solved for two different materials.

If the boundary value problem (1.10), (1.14) and (1.12) is solved, then in addition to problem 1 of Sect.1 for a "smooth stamp-free boundary", the problem of a "coverplate-coupled stamp" is also solved for the material (3.20). The boundary value problem (1.10), (1.16) and (1.12) corresponds both to problem 2 of Sect.1 of the "cover plate-free boundary" and to the problem of a "smooth stamp-coupled stamp" for the material (3.20).

In the special case of an elastic incompressible material $m = \mu = 1$, the governing relationships (1.1) and (3.20) are identical. Eqs.(1.10) and (3.12) have identical form (the notation of Sect.3)

$$\psi'''' + 2(t^2 - 2t + 2)\psi'' + t^2(2-t)^2\psi = 0 \tag{3.22}$$

The boundary conditions agree in pairs. The analogy becomes still more complete.

Combining the conditions of the designated four types differently at the different wedge faces, ten different problems can be obtained. Three of them, studied in Sects.1 and 2 are encountered most often. The same applies for the constraint $m \geqslant 1$ on the form of the governing relationships. The analogy formulated enables one to halve the amount of work required to investigate the general case for arbitrary m. It is still more important that the approach based on utilizing the stream function is carried over practically without change to the axisymmetric problems of a stamp and a cone. Construction of the asymptotic form near the edge for the solution of the axisymmetric problem of a stamp would enable us to estimate, in particular, the influence of tension-compression of near-lying fibres parallel to the edge on the local fields.

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